INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS
Int. J. Numer. Meth. Fluids 2001; **35**: 495–518

A non-iterative numerical approach for two-dimensional viscous flow problems governed by the Falkner–Skan equation

Shi-Jun Liao*,[1]

School of Naval Architecture and Ocean Engineering, Shanghai Jiao Tong University, Shanghai 200030, China

SUMMARY

In this paper, a non-iterative numerical approach for two-dimensional laminar viscous flow over a semi-infinite flat plane, governed by the Falkner–Skan equation is proposed. This approach can solve the non-linear Falkner–Skan equation without any iteration and verifies that a direct numerical approach could be proposed even for non-linear problems. Furthermore, this approach can also provide a family of iterative formulae, so that it logically contains traditional iterative techniques. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: direct numerical technique; Falkner–Skan equation; strong non-linearity; viscous flow

1. INTRODUCTION

Even with using high-performance computers, it is generally not easy to solve strongly non-linear problems. Traditionally, the iterative methodology is unavoidable for most numerical approaches to non-linear problems and more conditions must be satisfied to ensure the iterative process is convergent. Most traditional iterative techniques provide little freedom to select those features that are critical and important for the convergence of a related iterative process. So, it seems beneficial to propose an approach to non-linear problems that can give more freedom.

Liao [1–4] suggested a new kind of analytical technique for non-linear problems, namely the homotopy analysis method (HAM). Based on homotopy of topology, the validity of the HAM is independent of whether or not there exist small parameters in the non-linear equations considered, so that the HAM can overcome the limitations and restrictions of perturbation techniques. Although the HAM is an analytical technique, its basic ideas can be applied to

* Correspondence to: School of Naval Architecture and Ocean Engineering, Shanghai Jiao Tong University, 1954 Huashan Road, Shanghai 200030, China.
[1] E-mail: sjliao@mail.sjtu.edu.cn

Received November 1998
Revised March 1999

develop new numerical techniques. For example, Liao [5,6] and Liao and Chwang [7] successfully applied the HAM to propose a so-called general boundary element method (GBEM), which is valid even for those non-linear problems whose governing equations do not contain any linear terms (note that in this case the traditional BEM is invalid because no fundamental solution exists). Liao [8] further pointed out that similar approaches are also suitable for other numerical techniques, such as the finite difference method, finite element method, and so on. In this paper, the basic ideas of the HAM are applied to propose a non-iterative numerical approach for two-dimensional viscous flow problems governed by the non-linear Falkner–Skan equation. It will be shown that the proposed approach can solve non-linear differential equations without iteration. Furthermore, the proposed approach can also provide a family of iterative formulae, so that it contains, in logic, the most traditional iterative techniques.

2. BASIC IDEAS

2.1. Mathematical formulae

Laminar viscous flow over a two-dimensional semi-infinite plate is governed by the Falkner–Skan [9,10] equation

$$F'''(\eta) + \alpha F(\eta)F''(\eta) + \beta\{1 - [F'(\eta)]^2\} = 0, \quad \eta \in [0, +\infty) \tag{2.1}$$

with boundary conditions

$$F(0) = F'(0) = 0, \quad F'(+\infty) = 1 \tag{2.2}$$

Notice that Equation (2.1) is defined in an infinite domain $\eta \in [0, +\infty)$. In order to overcome this difficulty, the following transformation is made

$$\phi(\tau) = F'(\eta), \quad \tau = \frac{\sqrt{F}}{1 + \sqrt{F}} \tag{2.3}$$

so that Equations (2.1) and (2.2) become

$$\tau(1-\tau)^6\phi\left[\phi\left(\frac{d^2\phi}{d\tau^2}\right) + \left(\frac{d\phi}{d\tau}\right)^2\right] + (1-\tau)[2\alpha\tau^4 - (1+2\tau)(1-\tau)^4\phi]\phi\,\frac{d\phi}{d\tau} + 4\beta\tau^3(1-\phi^2) = 0, \quad \tau \in [0, 1] \tag{2.4}$$

and

$$\phi(0) = 0, \qquad \phi(1) = 1 \tag{2.5}$$

respectively. Note that the governing equation (2.4) does not contain any linear terms in the unknown function $\phi(\tau)$, thus its non-linearity is rather strong.

Let $\mathscr{A}$ denote a non-linear differential operator defined by

$$\mathscr{A}\phi = \tau(1-\tau)^6\phi\left[\phi\left(\frac{d^2\phi}{d\tau^2}\right)+\left(\frac{d\phi}{d\tau}\right)^2\right]+(1-\tau)[2\alpha\tau^4-(1+2\tau)(1-\tau)^4\phi]\phi\frac{d\phi}{d\tau} + 4\beta\tau^3(1-\phi^2) \tag{2.6}$$

Let $\hbar \neq 0$, $p \in [0, 1]$ be real numbers. To apply the HAM, first of all, the so-called zeroth-order deformation equation is constructed,

$$(1-p)\{\mathscr{L}[\Phi(\tau,p,\hbar)]-\mathscr{L}[\phi_0(\tau)]\} = p\hbar\mathscr{A}[\Phi(\tau,p,\hbar)], \quad \tau\in[0,1], \quad p\in[0,1], \quad \hbar\neq 0 \tag{2.7}$$

with boundary conditions

$$\Phi(0,p,\hbar)=0, \qquad \Phi(1,p,\hbar)=1 \tag{2.8}$$

where the auxiliary operator $\mathscr{L}$ is a linear second-order ordinary differential operator satisfying

$$\mathscr{L}\Phi = 0 \Leftrightarrow \Phi = 0 \tag{2.9}$$

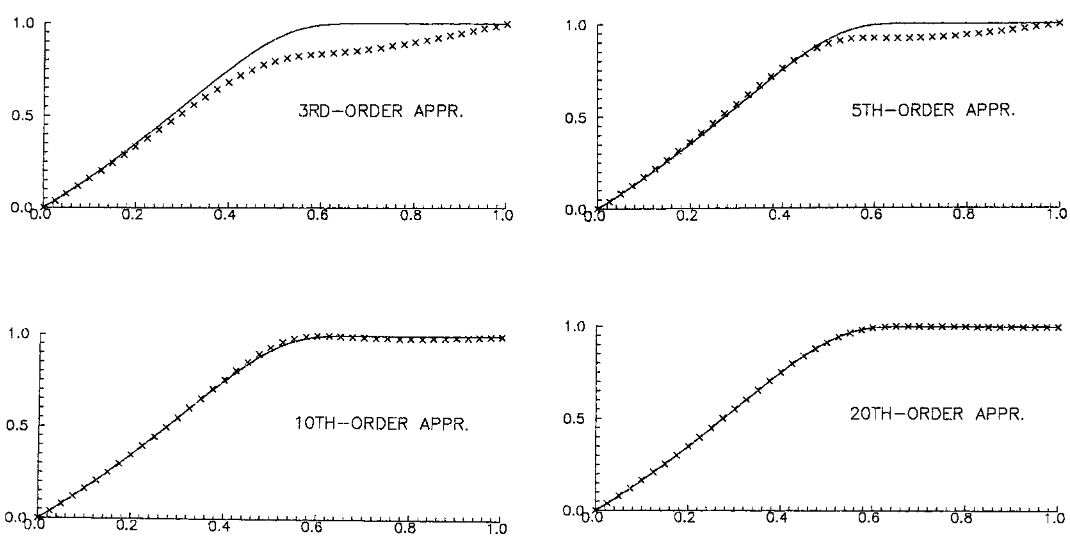

Figure 1. Approximations in the case of $\alpha = \frac{1}{2}$, $\beta = 1$, $\hbar = -0.1$ and $\mathscr{L} = \mathscr{L}_1$; $\times$, approximate results by means of no iterations; —, exact solution; horizontal axis, τ; vertical axis, $\phi(\tau)$.

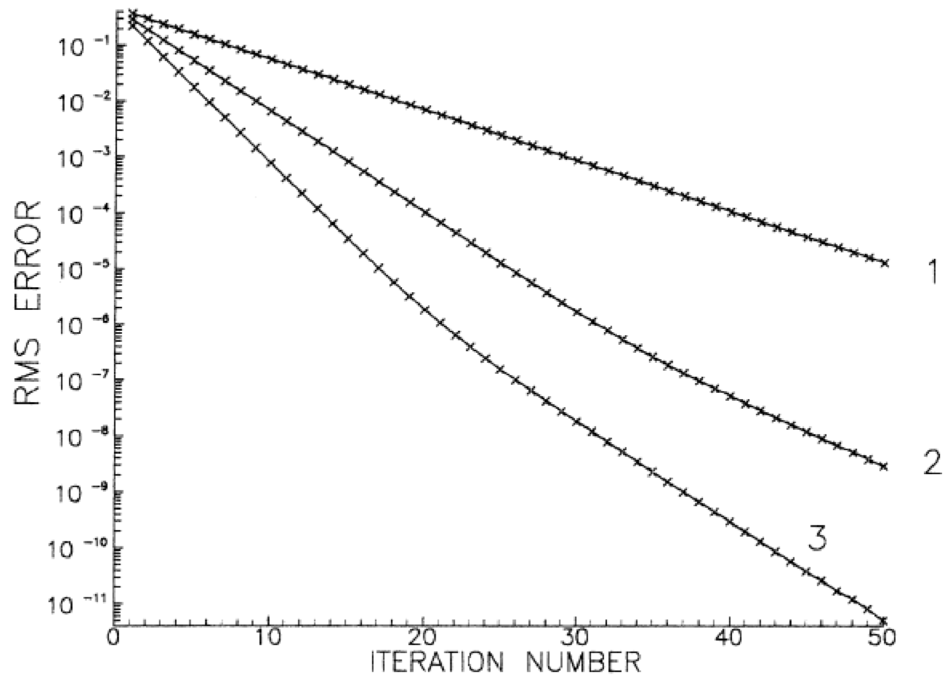

Figure 2. R.m.s. errors of the iterative process when $\alpha = \frac{1}{2}$, $\beta = 1$ and $\mathscr{L} = \mathscr{L}_1$; curve 1, r.m.s. for first-order iterative formula (3.11) ($M = 1$); curve 2: r.m.s. for second-order iterative formula (3.11) ($M = 2$); curve 3: r.m.s. for third-order iterative formula (3.11) ($M = 3$).

and $\phi_0(\tau)$ is an initial approximation satisfying the boundary conditions $\phi_0(0) = 0$ and $\phi_0(1) = 1$. Clearly, $\phi_0(\tau)$ and $\phi(\tau)$ are solutions of (2.7) and (2.8) when $p = 0$ and $p = 1$ respectively. Thus, we have

$$\Phi(\tau, 0, \hbar) = \phi_0(\tau) \tag{2.10}$$

$$\Phi(\tau, 1, \hbar) = \phi(\tau) \tag{2.11}$$

Write

$$\phi_0^{[k]}(\tau, \hbar) = \left. \frac{\partial^k \Phi(\tau, p, \hbar)}{\partial p^k} \right|_{p=0} \tag{2.12}$$

namely the kth-order deformation derivatives of $\Phi(\tau, p, \hbar)$. Then, using (2.10), the Maclaurin series of $\Phi(\tau, p, \hbar)$ about p is

$$\Phi(\tau, p, \hbar) = \Phi(\tau, 0, \hbar) + \sum_{k=1}^{+\infty} \frac{\phi_0^{[k]}(\tau, \hbar)}{k!} p^k = \phi_0(\tau) + \sum_{k=1}^{+\infty} \frac{\phi_0^{[k]}(\tau, \hbar)}{k!} p^k \tag{2.13}$$

If $\phi_0(\tau)$, $\hbar$ and the auxiliary linear operator $\mathscr{L}$ are so selected that the above Maclaurin series is convergent at $p = 1$ in the range of $\tau \in [0, 1]$, by Equation (2.11) one has that

$$\phi(\tau) = \phi_0(\tau) + \sum_{k=1}^{+\infty} \frac{\phi_0^{[k]}(\tau, \hbar)}{k!} \tag{2.14}$$

Differentiating the zeroth-order deformation equations (2.7) and (2.8) k times with respect to p and then setting $p = 0$, we obtain the so-called kth-order deformation equations governing $\phi_0^{[k]}(\tau, \hbar)$ $(k \geq 1)$,

$$\mathscr{L}\phi_0^{[k]} = f_k(\tau, \hbar), \qquad \tau \in [0, 1], \quad \hbar \neq 0 \tag{2.15}$$

with boundary conditions

$$\phi_0^{[k]}(0, \hbar) = 0, \qquad \phi_0^{[k]}(1, \hbar) = 0 \tag{2.16}$$

where

$$f_1(\tau, \hbar) = \hbar \mathscr{A} \phi_0 \tag{2.17}$$

and

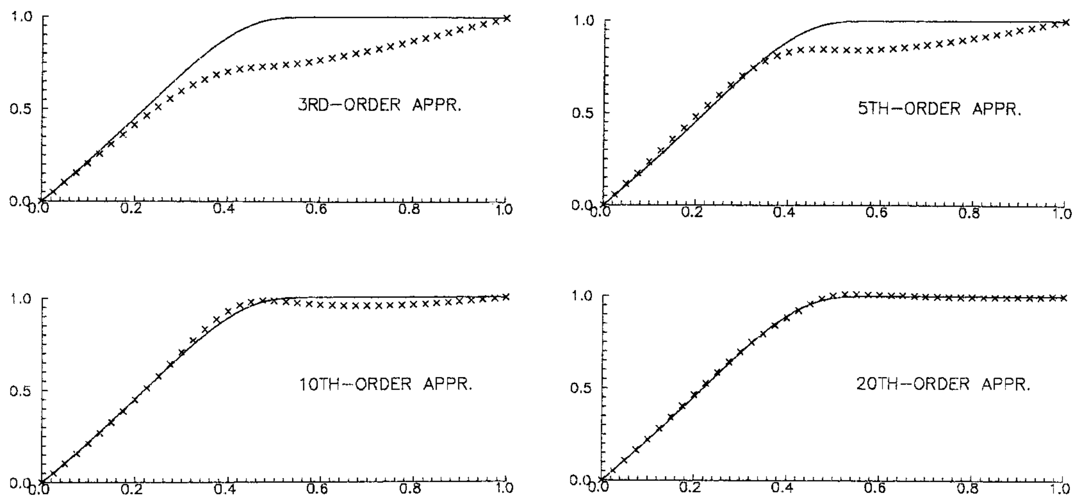

Figure 3. Approximations in the case of $\alpha = 3$, $\beta = 3$, $\hbar = -0.075$ and $\mathscr{L} = \mathscr{L}_1$; $\times$, approximate results by means of no iterations; —, exact solution; horizontal axis, τ; vertical axis, $\phi(\tau)$.

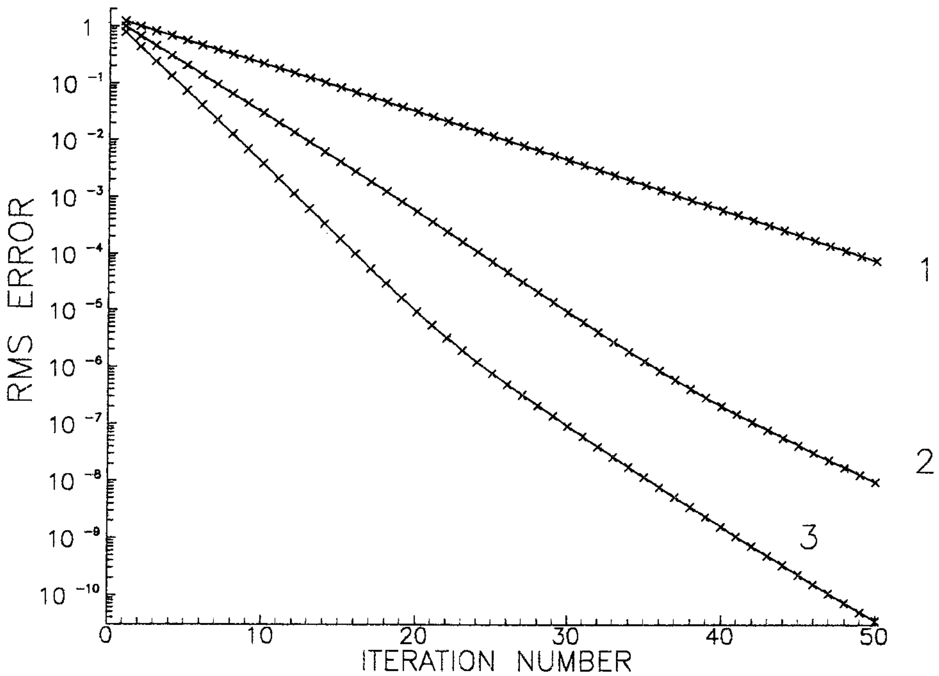

Figure 4. R.m.s. errors of the iterative process when $\alpha = \beta = 3$ and $\mathscr{L} = \mathscr{L}_1$; curve 1, r.m.s. for first-order iterative formula (3.11) ($M = 1$); curve 2, r.m.s. for second-order iterative formula (3.11) ($M = 2$); curve 3, r.m.s. for third-order iterative formula (3.11) ($M = 3$).

$$f_k(\tau, \hbar) = k\left\{\mathscr{L}\phi_0^{[k-1]} + \hbar\,\frac{\mathrm{d}^{k-1}\mathscr{A}[\Phi(\tau, p, \hbar)]}{\mathrm{d}p^{k-1}}\bigg|_{p=0}\right\}, \quad k \geq 2 \tag{2.18}$$

It is emphasized that (2.15) is a *linear* differential equation with a linear boundary condition (2.16), which can be easily solved.

2.2. Convergence of the series of approximations

Liao [3] generally proved that, if the series of approximations given by the HAM is convergent, it must converge to one solution of the non-linear problem under consideration. Similarly, it can be proven in this paper that, if the series

$$\phi_0(\tau) + \sum_{k=1}^{+\infty} \frac{\phi_0^{[k]}(\tau, \hbar)}{k!} \tag{2.19}$$

converges, it must be a solution of Equations (2.4) and (2.5).

Note that the initial approximation $\phi_0(\tau)$ satisfies the boundary conditions (2.5). Thus, according to Equation (2.16), the infinite series (2.19) converges to 0 at $\tau = 0$ and to 1 at $\tau = 1$

respectively. Clearly, $\mathscr{A}\phi_0(\tau)$ denotes the residual error of the governing equation (2.4) at $p=0$, i.e. under the initial approximation $\phi_0(\tau)=\Phi(\tau, 0, \hbar)$. In general,

$$\mathscr{R}(\tau, p)=\mathscr{A}[\Phi(\tau, p, \hbar)] \tag{2.20}$$

denotes the residual error of (2.4) at $p\in[0, 1]$ and $\tau\in[0, 1]$. Clearly, if $\Phi(\tau, 1, \hbar)$ is a solution of (2.4), it holds that

$$\mathscr{R}(\tau, 1)=\mathscr{A}[\Phi(\tau, 1, \hbar)]=0, \quad \tau\in[0, 1] \tag{2.21}$$

Notice that the Maclaurin series of $\mathscr{R}(\tau, p)$ is

$$\mathscr{A}\phi_0(\tau)+\sum_{k=1}^{+\infty}\left.\frac{\mathrm{d}^k\mathscr{A}[\Phi(\tau, p, \hbar)]}{\mathrm{d}p^k}\right|_{p=0} p^k \tag{2.22}$$

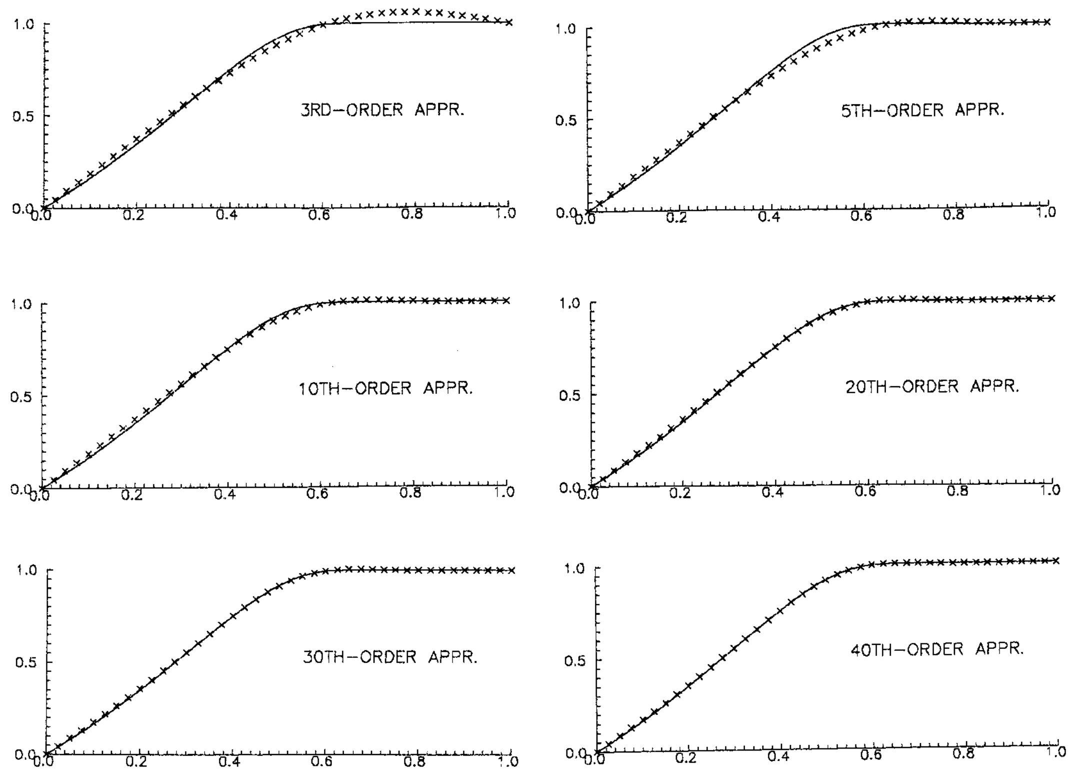

Figure 5. Approximations in the case of $\alpha=\frac{1}{2}$, $\beta=1$, $\hbar=-5$ and $\mathscr{L}=\mathscr{L}_2$ $(\gamma=1)$; $\times$, approximate results by means of no iterations; —, exact solution; horizontal axis, τ; vertical axis, $\phi(\tau)$.

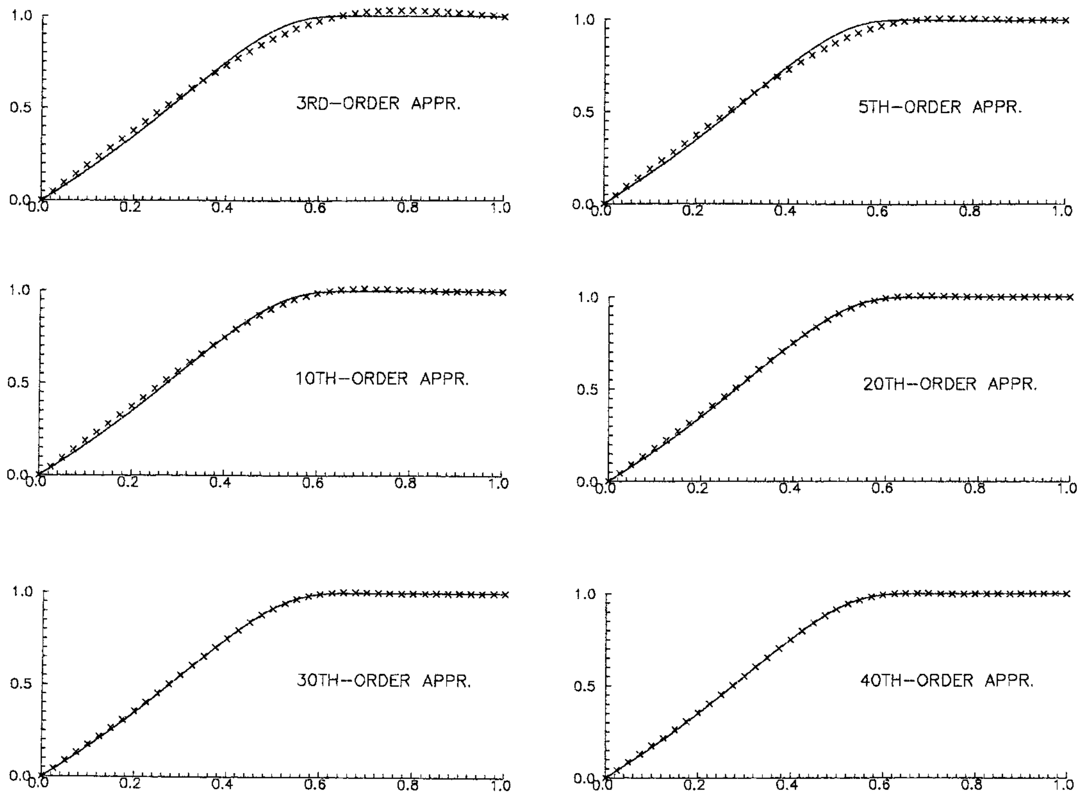

Figure 6. Approximations in the case of $\alpha = \frac{1}{2}$, $\beta = 1$, $\hbar = -5$ and $\mathscr{L} = \mathscr{L}_2$ ($\gamma = 3$); $\times$, approximate results by means of no iterations; —, exact solution; horizontal axis, τ; vertical axis, $\phi(\tau)$.

When $p = 1$, the above Maclaurin series gives

$$\mathscr{A}\phi_0(\tau) + \sum_{k=1}^{+\infty} \frac{\mathrm{d}^k \mathscr{A}[\Phi(\tau, p, \hbar)]}{\mathrm{d}p^k}\bigg|_{p=0} \tag{2.23}$$

Hence, we need to prove that

$$\mathscr{A}\phi_0 + \sum_{k=1}^{+\infty} \frac{\mathrm{d}^k \mathscr{A}[\Phi(\tau, p, \hbar)]}{\mathrm{d}p^k}\bigg|_{p=0} = 0, \quad \tau \in [0, 1] \tag{2.24}$$

To prove (2.24), we have by (2.17), (2.18) and straightforward calculations that

$$f_k(\tau, \hbar) = \hbar k! \sum_{i=0}^{k-1} \frac{\mathrm{d}^i \mathscr{A}[\Phi(\tau, p, \hbar)]}{\mathrm{d}p^i}\bigg|_{p=0}, \quad k \geq 1 \tag{2.25}$$

Besides, if the series (2.19) is convergent, it holds that

$$\lim_{k \to +\infty} \frac{\phi_0^{[k]}(\tau, \hbar)}{k!} = 0 \tag{2.26}$$

Using (2.26), (2.15), (2.25) and (2.9), we have

$$\lim_{k \to +\infty} \mathscr{L}\left(\frac{\phi_0^{[k]}}{k!}\right) = \lim_{k \to +\infty} \frac{f_k(\tau, \hbar)}{k!} = \hbar \lim_{k \to +\infty} \sum_{i=0}^{k-1} \frac{\mathrm{d}^i \mathscr{A}[\Phi(\tau, p, \hbar)]}{\mathrm{d}p^i}\bigg|_{p=0} = 0 \tag{2.27}$$

Because we define $\hbar \neq 0$ in (2.7), we can obtain (2.24) from the above expression. Therefore, as long as the series (2.19) is convergent, it must converge to the solution of (2.4) and (2.5). This completes the proof.

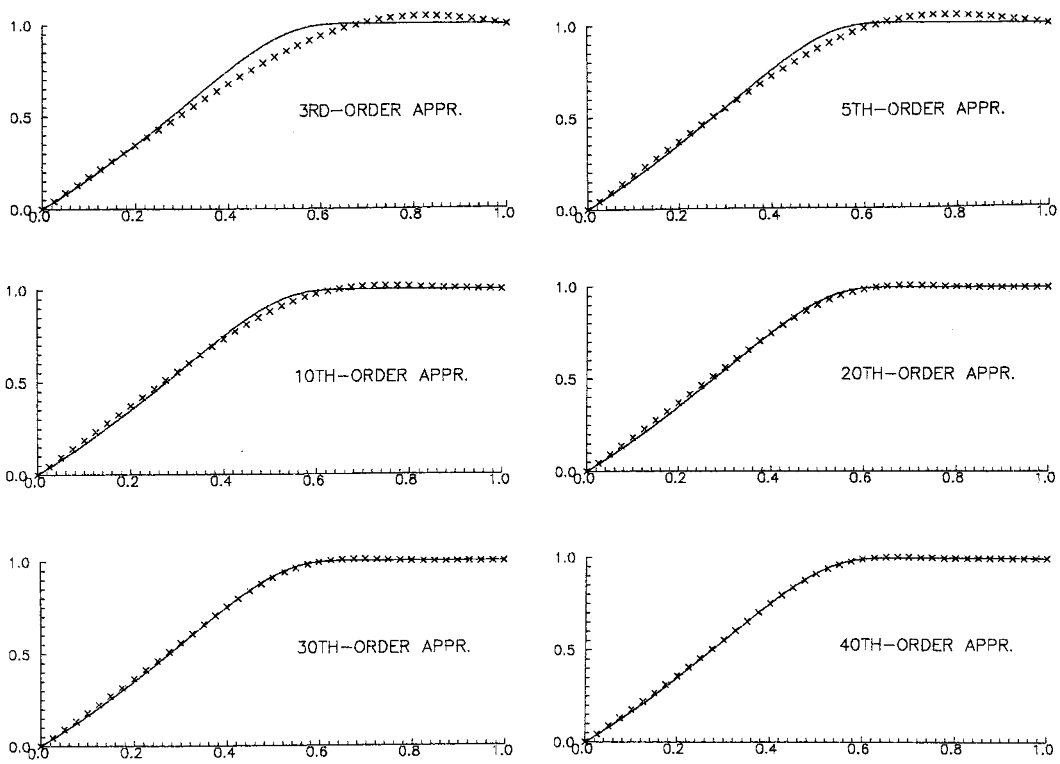

Figure 7. Approximations in the case of $\alpha = \frac{1}{2}$, $\beta = 1$, $\hbar = -3$ and $\mathscr{L} = \mathscr{L}_2$ ($\gamma = 1$); $\times$, approximate results by means of no iterations; —, exact solution; horizontal axis, τ; vertical axis, $\phi(\tau)$.

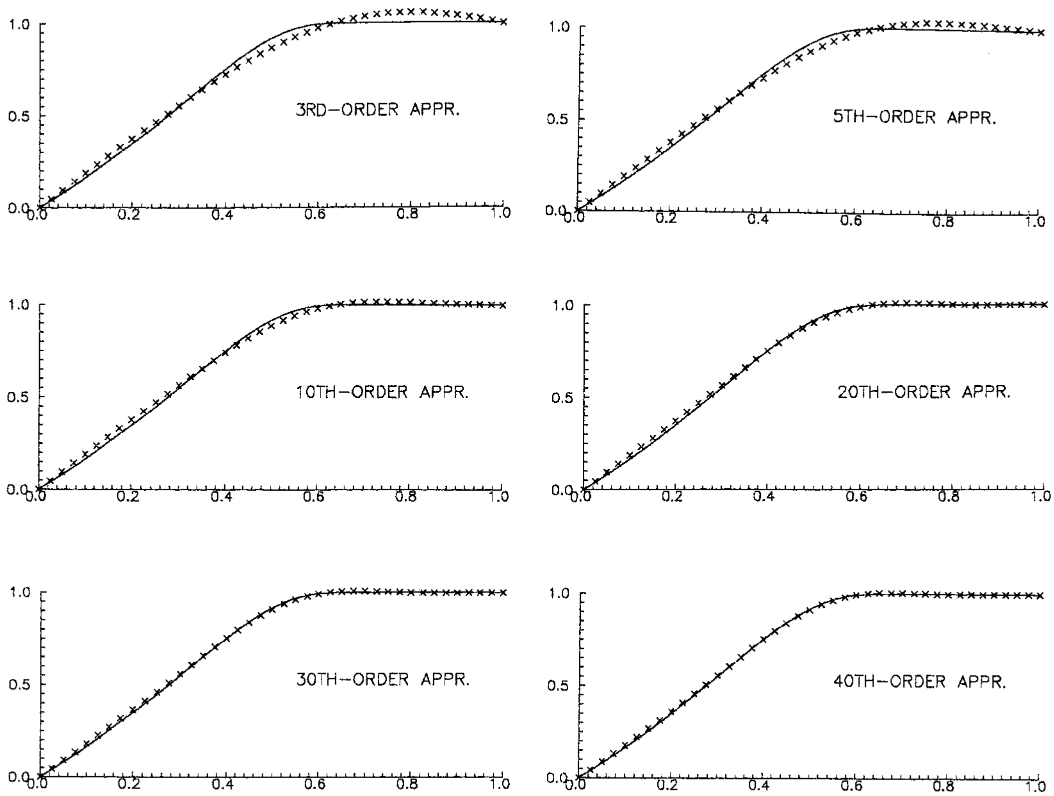

Figure 8. Approximations in the case of $\alpha = \frac{1}{2}$, $\beta = 1$, $\hbar = -3$ and $\mathscr{L} = \mathscr{L}_2$ ($\gamma = 3$); $\times$, approximate results by means of no iterations; —, exact solution; horizontal axis, τ; vertical axis, $\phi(\tau)$.

3. NUMERICAL CALCULATIONS

The convergence of series (2.19) depends on the initial approximation $\phi_0(\tau)$, the auxiliary parameter $\hbar$ and the auxiliary linear operator $\mathscr{L}$. As mentioned in References [1–8], there is a lot of freedom in their selection, and this increases the possibility of ensuring that series (2.19) is convergent. For the sake of simplicity, we simply select $\phi_0(\tau) = \tau$ and use an auxiliary linear operator in the following form:

$$\mathscr{L} = a(\tau)\frac{\partial^2}{\partial\tau^2} + b(\tau)\frac{\partial}{\partial\tau} + c(\tau) \tag{3.1}$$

where $a(\tau)$, $b(\tau)$, $c(\tau)$ are real functions. Consider here three kinds of auxiliary linear operators. First of all, one can use the initial approximation $\phi_0(\tau)$ to linearize the non-linear Equation (2.4). To do so, (2.4) is written in the form

$$A(\tau)\phi''(\tau) + B(\tau)\phi'(\tau) + C(\tau)\phi(\tau) + 4\beta\tau^3 = 0 \tag{3.2}$$

where

$$A(\tau) = \tau(1-\tau)^6\phi_0^2(\tau) \tag{3.3}$$

$$B(\tau) = \tau(1-\tau)^6\phi_0(\tau)\phi_0(\tau) + (1-\tau)\,\lfloor 2\alpha\tau^4 - (1-\tau)^4(1+2\tau)\phi_0(\tau)\rfloor\,\phi_0(\tau) \tag{3.4}$$

$$C(\tau) = -4\beta\tau^3\phi_0(\tau) \tag{3.5}$$

To satisfy (2.9), the first auxiliary linear operator is defined as follows:

$$\mathscr{L}_1\phi(\tau) = A(\tau)\phi''(\tau) + B(\tau)\phi'(\tau) + C(\tau)\phi(\tau) \tag{3.6}$$

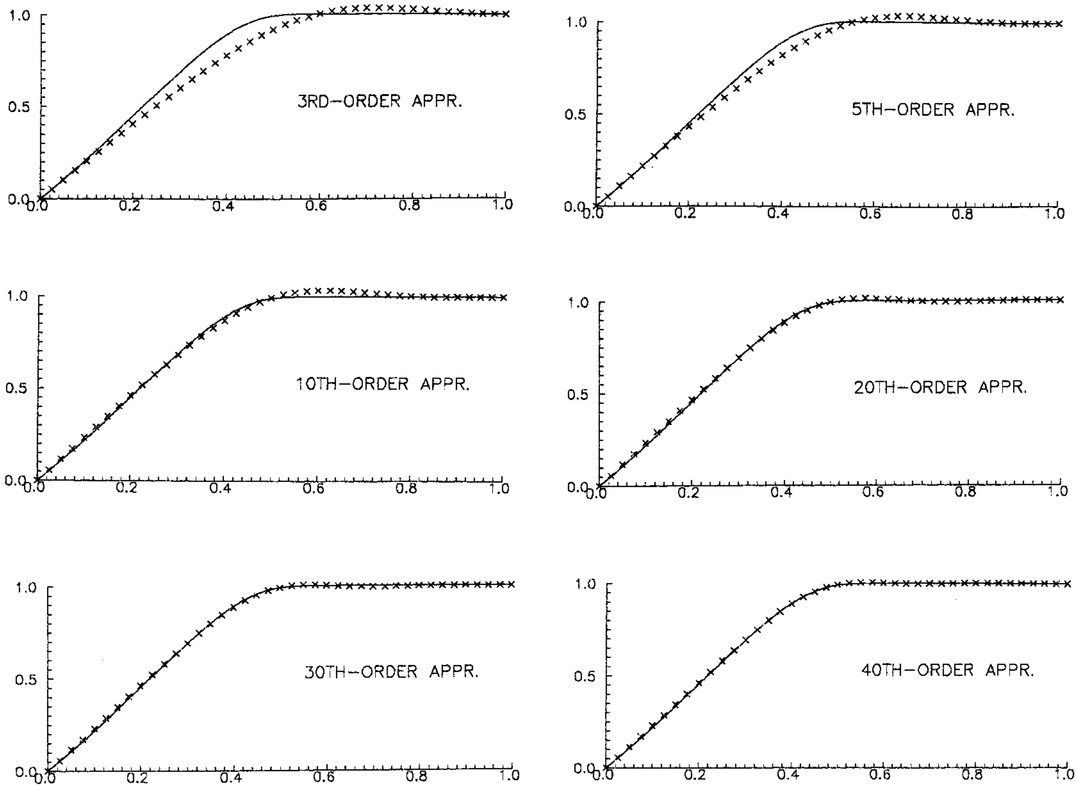

Figure 9. Approximations in the case of $\alpha = \beta = 3$, $\hbar = -2$ and $\mathscr{L} = \mathscr{L}_2$ ($\gamma = 2$); ×, approximate results by means of no iterations; —, exact solution; horizontal axis, τ; vertical axis, $\phi(\tau)$.

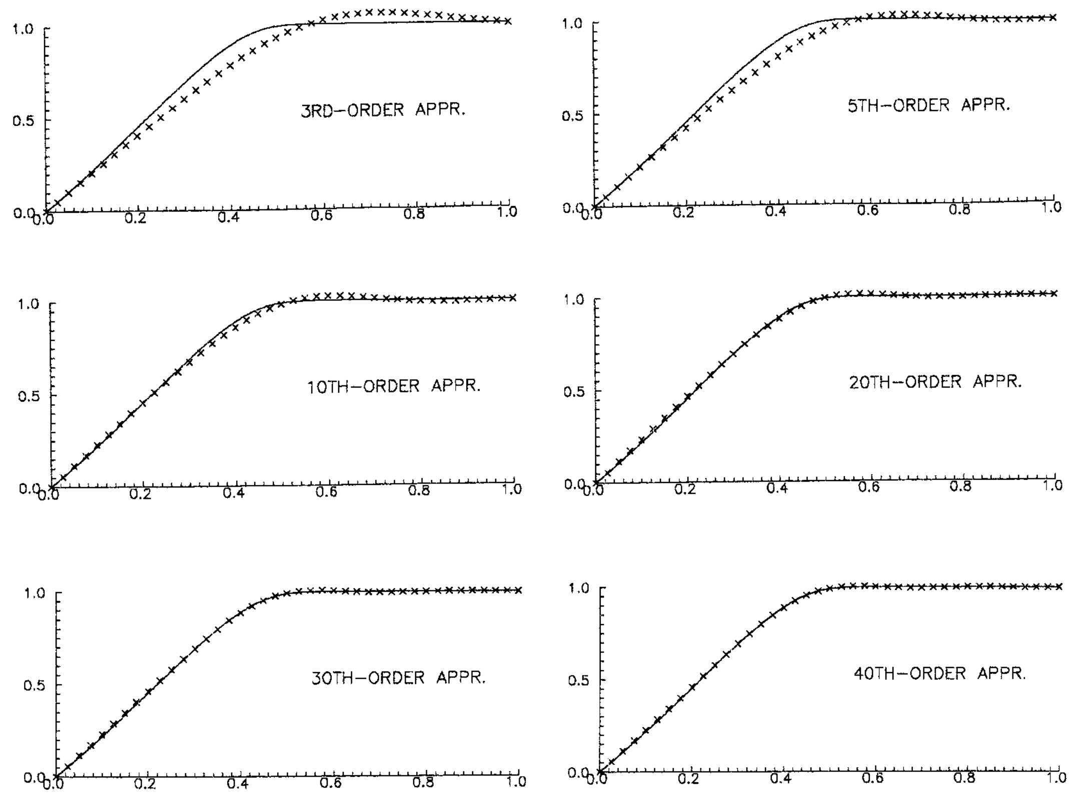

Figure 10. Approximations in the case of $\alpha = \beta = 3$, $\hbar = -2$ and $\mathscr{L} = \mathscr{L}_2$ ($\gamma = 1$); $\times$, approximate results by means of no iterations; —, exact solution; horizontal axis, τ; vertical axis, $\phi(\tau)$.

corresponding to $a(\tau) = A(\tau)$, $b(\tau) = B(\tau)$ and $c(\tau) = C(\tau)$. Also,

$$\mathscr{L}_2\phi(\tau) = \phi''(\tau) + \gamma\phi(\tau), \quad \gamma > 0 \tag{3.7}$$

and

$$\mathscr{L}_3\phi(\tau) = \phi''(\tau) - \gamma\phi(\tau), \quad \gamma > 0 \tag{3.8}$$

are selected as the second and third auxiliary linear operators, corresponding to $a(\tau) = 1$, $b(\tau) = 0$ and $c(\tau) = \pm\gamma$ respectively. Without loss of generality, the finite difference method (FDM) is used to solve the linear differential equations (2.15) and (2.16). For the numerical computation, the region [0, 1] is divided into N ($N = 1000$) equal sub-domains, say, $\tau_m = m\Delta\tau = m/N$ ($0 \leq m \leq N$). Then, applying the FDM to (2.16) and (2.15), the following set of linear algebraic equations is obtained ($k \geq 1$):

$$\phi_0^{[k]}(0, \hbar) = \phi_0^{[k]}(1, \hbar) = 0 \tag{3.9}$$

$$\left[a(\tau_m) - \frac{b(\tau_m)\Delta\tau}{2}\right]\phi_0^{[k]}(\tau_{m-1}, \hbar) - [2a(\tau_m) - c(\tau_m)(\Delta\tau)^2]\phi_0^{[k]}(\tau_m, \hbar)$$
$$+\left[a(\tau_m) + \frac{b(\tau_m)\Delta\tau}{2}\right]\phi_0^{[k]}(\tau_{m+1}, \hbar) = f_k(\tau_m, \hbar)(\Delta\tau)^2, \quad 1 \leq m \leq N-1 \tag{3.10}$$

which can be easily solved by the direct method. Here, it is emphasized that the related coefficient matrix **M** of the above discrete linear algebraic equations is constant for *all* $\phi_0^{[k]}$ $(k \geq 1)$, so that if its inverse matrix $\mathbf{M}^{-1}$ is obtained it can be used again and again to get all $\phi_0^{[k]}$ $(k \geq 1)$.

If series (2.19) converges, we have the Mth-order approximation

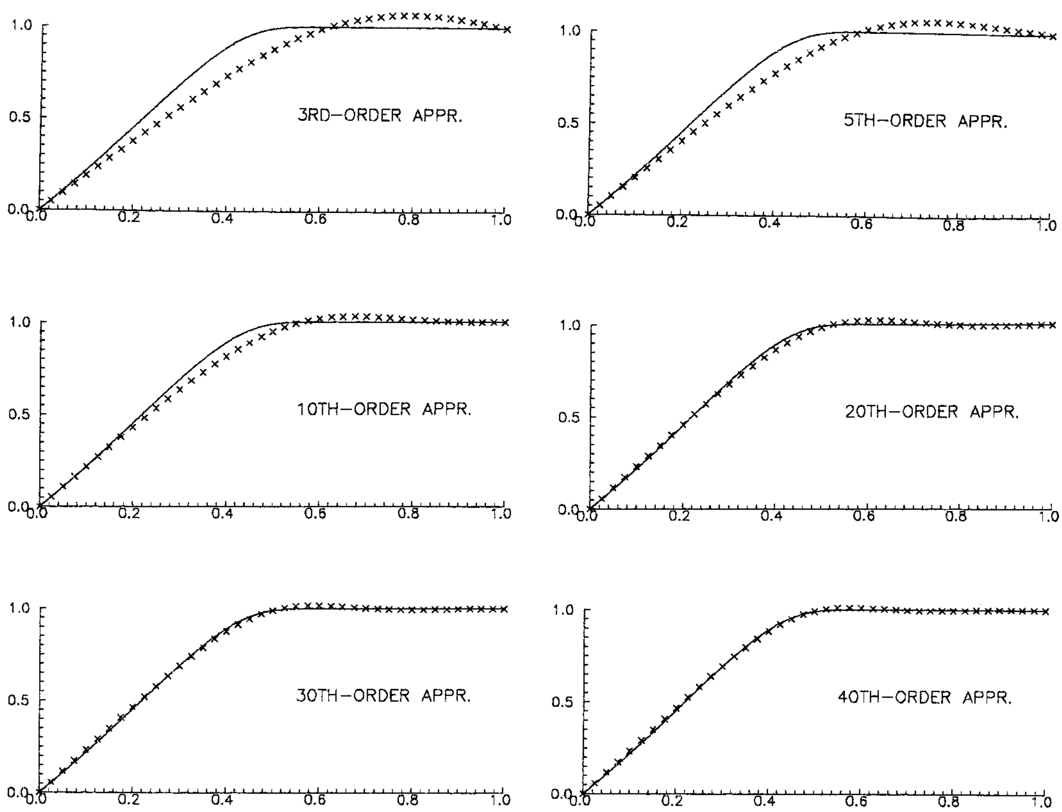

Figure 11. Approximations in the case of $\alpha = \beta = 3$, $\hbar = -1$ and $\mathscr{L} = \mathscr{L}_2$ $(\gamma = 2)$; $\times$, approximate results by means of no iterations; —, exact solution; horizontal axis, τ; vertical axis, $\phi(\tau)$.

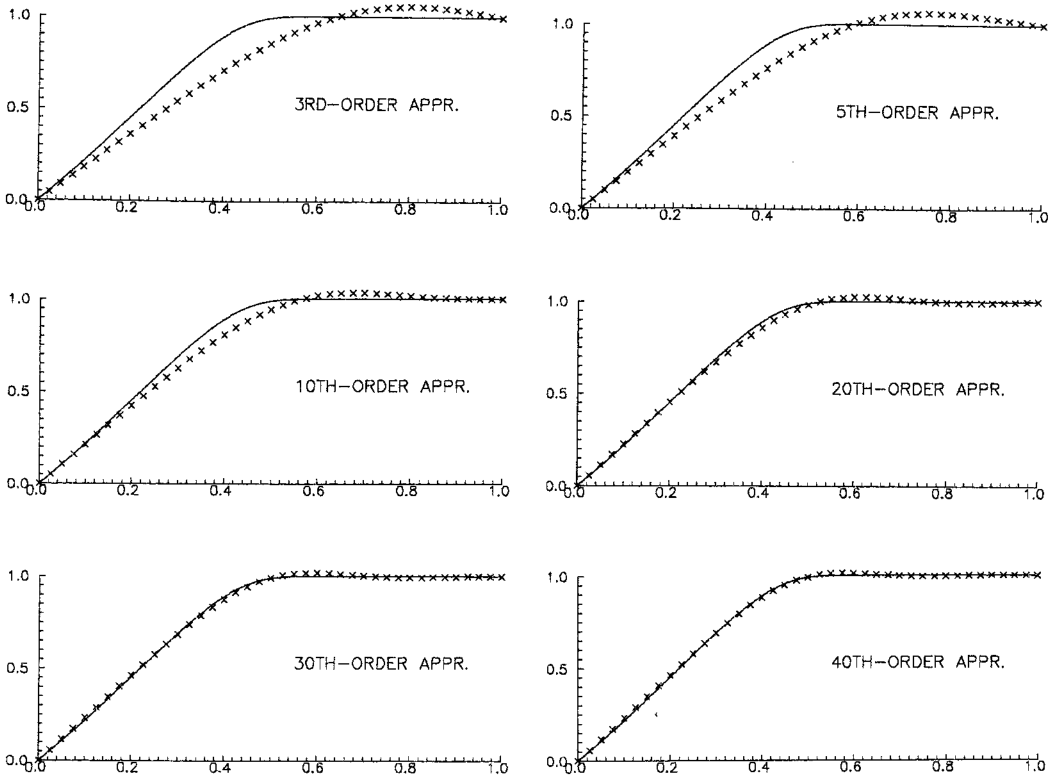

Figure 12. Approximations in the case of $\alpha = \beta = 3$, $\hbar = -1$ and $\mathscr{L} = \mathscr{L}_2$ $(\gamma = 1)$; $\times$, approximate results by means of no iterations; —, exact solution; horizontal axis, τ; vertical axis, $\phi(\tau)$.

$$\phi_M(\tau) = \phi_0(\tau) + \sum_{k=1}^{M} \frac{\phi_0^{[k]}(\tau, \hbar)}{k!} \tag{3.11}$$

The root-mean-square residual error of (2.4) under the above Mth-order approximation $\phi_M(\tau)$ is defined by

$$\mathscr{E}_M = \sqrt{\frac{\sum_{m=1}^{N-1} |\mathscr{A}\phi_M(\tau_m)|^2}{N-1}} \tag{3.12}$$

When series (2.19) is convergent and the order M is sufficiently high, Equation (3.11) can give an accurate enough approximation and therefore no iteration is necessary. This means that the approach here can give accurate enough solutions of a non-linear problem without iteration. On the other hand, if the order M is not high enough, the approximation $\phi_M(\tau)$ given by (3.11) might be unsatisfactory, which, however, can be further used as a new initial approximation $\phi_0(\tau)$ to get better approximations. Therefore, formula (3.11) also provides a family of iterative formulae in the parameter $\hbar$ and order M. It is easy to prove that when $M = 1$, and we consider $\omega = -\hbar$ as the iterative factor, (3.11) corresponds to the Gauss–Seidel iteration formula. Thus, the proposed approach logically contains the iterative methodology. In this paper, all of the so-called 'exact' solutions are given by this iterative approach.

To illustrate the validity and potential of the proposed numerical approach, without loss of generality, two cases are considered here. One is $\alpha = \frac{1}{2}$, $\beta = 1$, and the other is $\alpha = \beta = 3$.

3.1. $\alpha = \frac{1}{2}$, $\beta = 1$ and $\mathscr{L} = \mathscr{L}_1$

First of all, we use $\mathscr{L}_1$ defined by (3.6) as the auxiliary linear operator. As mentioned above, if series (2.19) converges and the order M of approximation is sufficiently large, an accurate enough approximation of the non-linear equation (2.4) can be obtained without any iteration. Our calculations show that this is indeed true. For instance, when $-0.15 \leq \hbar < 0$, the corresponding series (2.19) indeed converges to the exact solution of (2.4), as shown in Figure

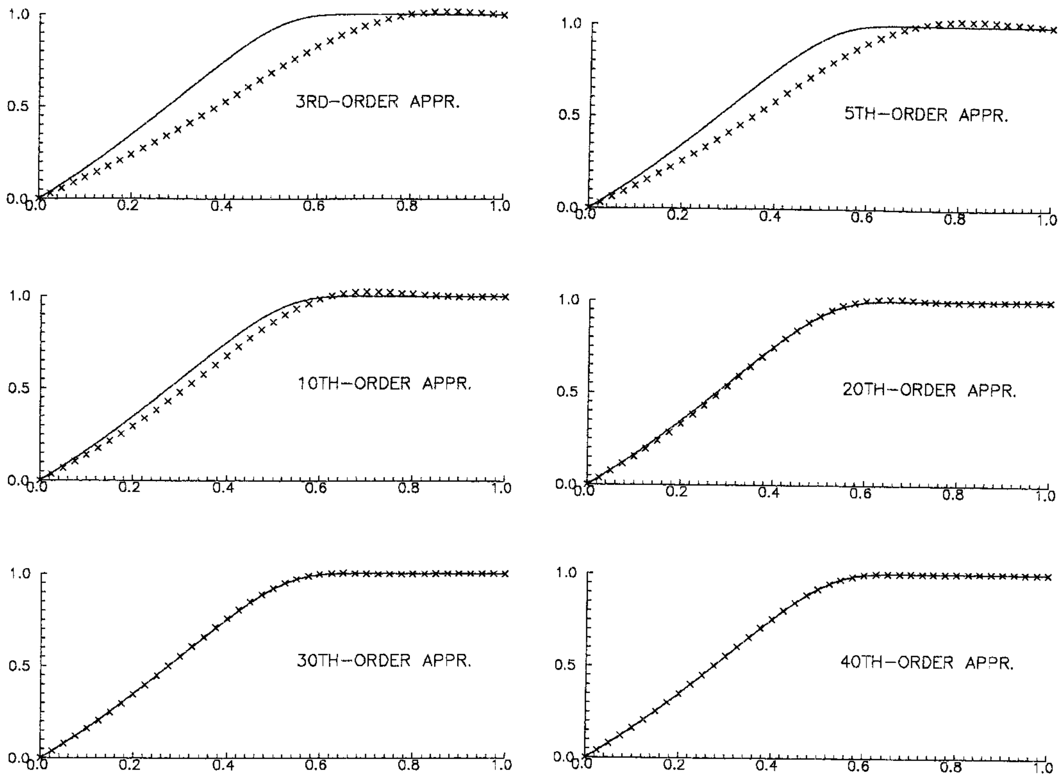

Figure 13. Approximations in the case of $\alpha = \frac{1}{2}$, $\beta = 1$, $\hbar = -10$ and $\mathscr{L} = \mathscr{L}_3$ ($\gamma = 50$); $\times$, approximate results by means of no iterations; —, exact solution; horizontal axis, τ; vertical axis, $\phi(\tau)$.

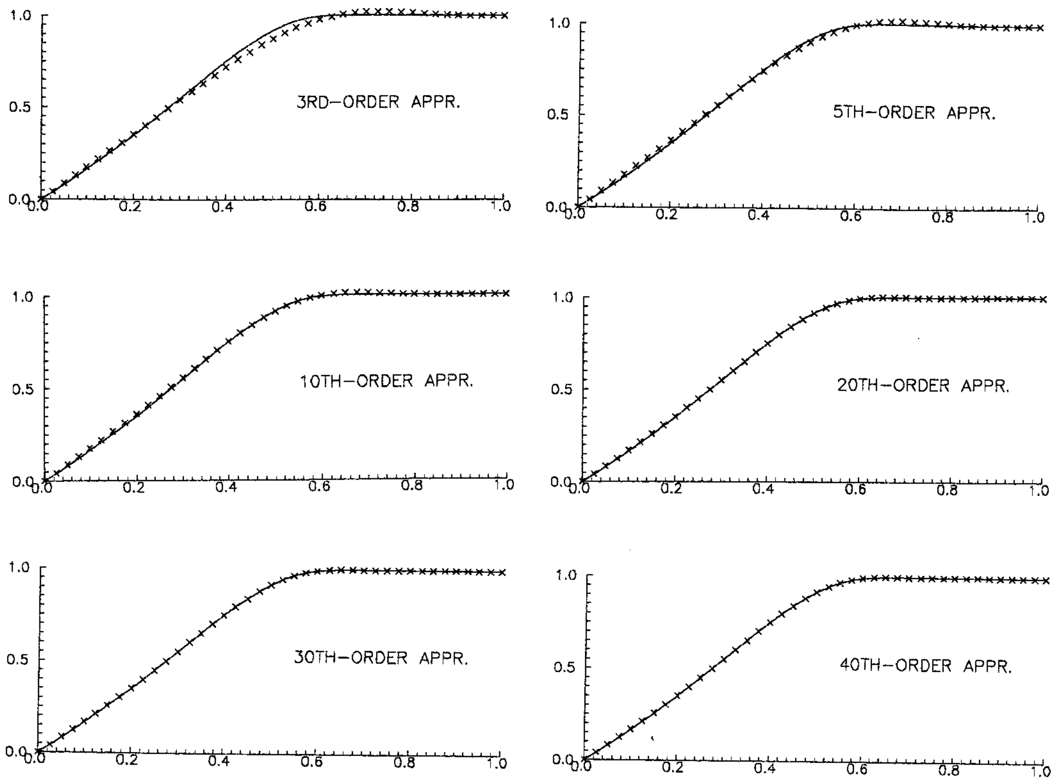

Figure 14. Approximations in the case of $\alpha = \frac{1}{2}$, $\beta = 1$, $\hbar = -10$ and $\mathscr{L} = \mathscr{L}_3$ ($\gamma = 5$); $\times$, approximate results by means of no iterations; —, exact solution; horizontal axis, τ; vertical axis, $\phi(\tau)$.

1 for the case $\hbar = -0.1$. In fact, when $\hbar = -0.1$, the tenth-order approximation given by (3.11) is accurate enough. Thus, an accurate enough approximation of a strongly non-linear problem can be indeed obtained with *no* iterations! Considering the governing place of the iterative methodology in numerical techniques for non-linear problems, this might increase our understanding about numerical approaches for non-linear equations. Note that, currently, more and more researchers like to apply direct methods to solve linear partial differential equations (for instance, refer to Greengard and Lee [6] and McKenney and Greengard [8]). It is interesting that the proposed approach can give an effective, direct numerical approach for non-linear problems. This might break some new fields of research.

The so-called 'exact solution' in Figure 1 is obtained by the above-mentioned iterative approach based on formula (3.11). The root-mean-square errors defined by (3.12) when using the first- ($M = 1$), second- ($M = 2$) and third-order ($M = 3$) iterative formulae of (3.11), in the case $\hbar = -0.1$, are shown in Figure 2. Clearly, the higher the order of the iterative formula (3.11), the faster the related iterative process converges. And when M is sufficiently large, no iteration is necessary. Similar qualitative conclusions have been given in References [5–8], so there is no need to discuss them here. For further details, please refer to Liao [5,6].

3.2. $\alpha = \beta = 3$ and $\mathscr{L} = \mathscr{L}_1$

Here, we use $\mathscr{L}_1$ defined by (3.6) as the auxiliary linear operator. The calculations show that, when $-0.075 \leq \hbar < 0$ and the order M is sufficiently high, (3.11) can give accurate enough approximations, as shown in Figure 3. Note that the non-linearity of (2.1) is now stronger than that in case of $\alpha = \frac{1}{2}$, $\beta = 1$. Even so, when $M \geq 20$, we still get an accurate enough approximation *without* iteration! Moreover, when $\hbar = -0.1$, the related root-mean-square errors of the iterative process based on formula (3.11) at the first- ($M = 1$), second- ($M = 2$) and third-order ($M = 3$) of approximation are shown in Figure 4. Once again, the same qualitative conclusion is obtained as that mentioned above, namely, the higher the order of the approximation, the faster the related iterative process converges. All of these indicate that the foregoing conclusions have general meanings.

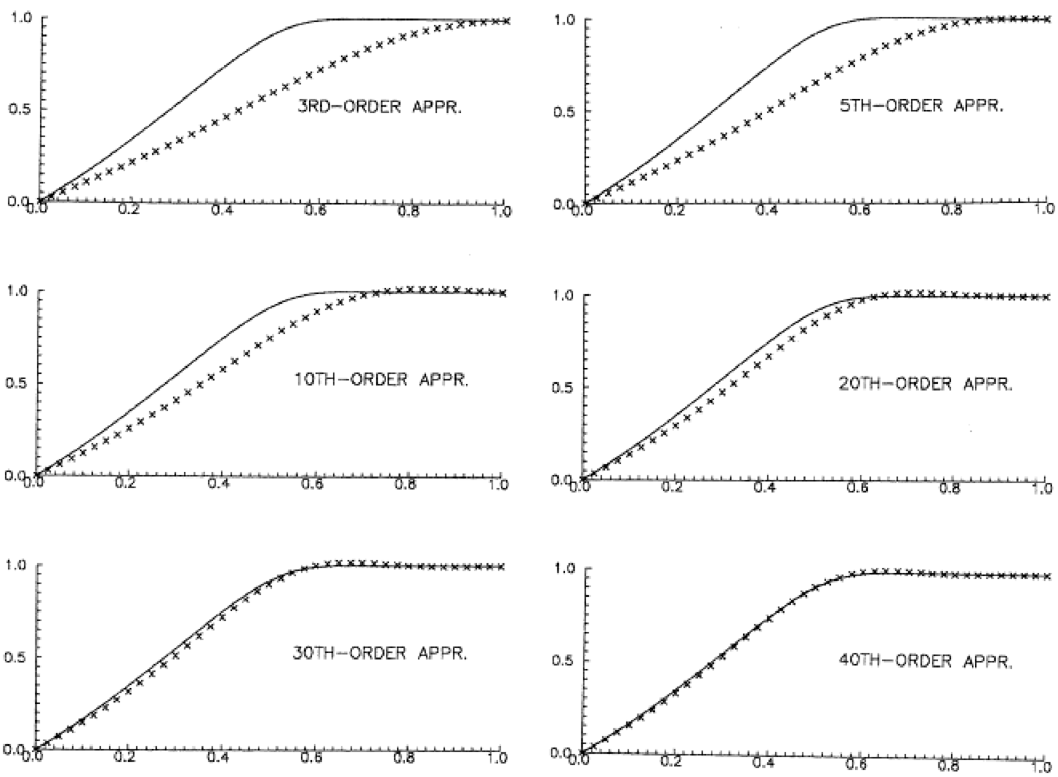

Figure 15. Approximations in the case of $\alpha = \frac{1}{2}$, $\beta = 1$, $\hbar = -5$ and $\mathscr{L} = \mathscr{L}_3$ ($\gamma = 50$); $\times$, approximate results by means of no iterations; —, exact solution; horizontal axis, τ; vertical axis, $\phi(\tau)$.

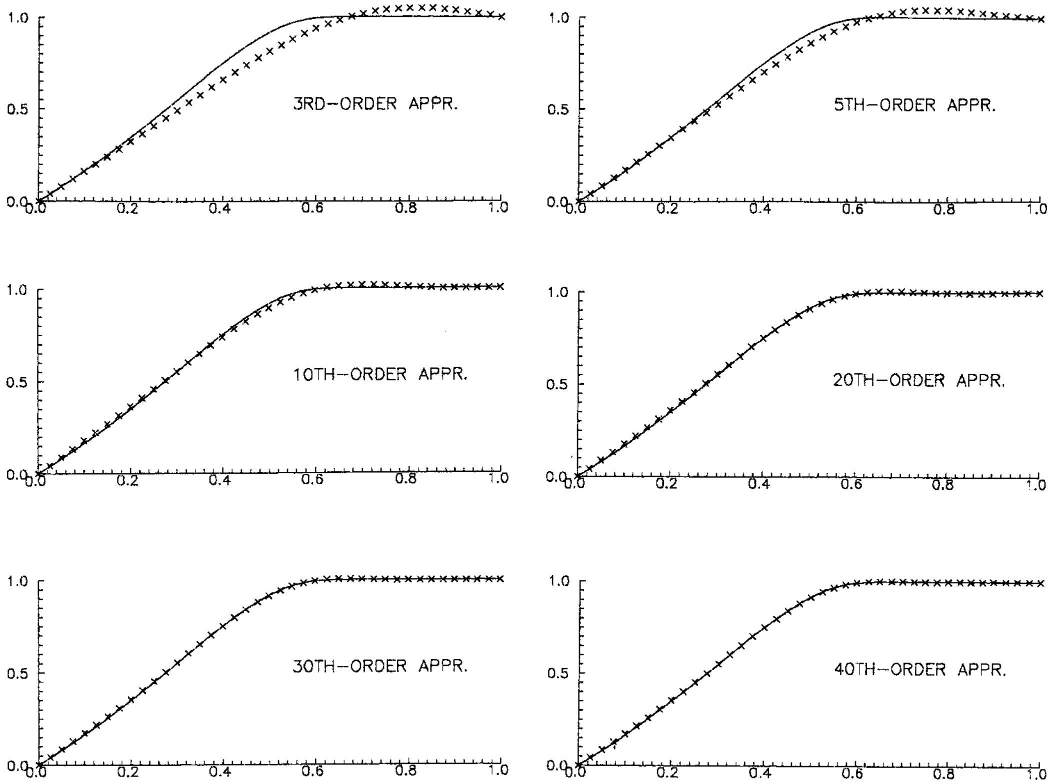

Figure 16. Approximations in the case of $\alpha = \frac{1}{2}$, $\beta = 1$, $\hbar = -5$ and $\mathscr{L} = \mathscr{L}_3$ ($\gamma = 5$); ×, approximate results by means of no iterations; —, exact solution; horizontal axis, τ; vertical axis, $\phi(\tau)$.

3.3. $\alpha = \frac{1}{2}$, $\beta = 1$ *and* $\mathscr{L} = \mathscr{L}_2$

Here we use $\mathscr{L}_2$ defined by (3.7) as the auxiliary linear operator. Our calculations indicate that, when $0 < \gamma \leq 3$ and $-5 \leq \hbar < 0$, formula (3.11) at a sufficiently high order of approximation can give accurate enough approximations so that no iteration is needed, as shown in Figures 5–8.

3.4. $\alpha = \beta = 3$ *and* $\mathscr{L} = \mathscr{L}_2$

Here we use $\mathscr{L}_2$ defined by (3.7) as the auxiliary linear operator. Our calculations indicate that, when $0 < \gamma \leq 2$ and $-2 \leq \hbar < 0$, formula (3.11) at a sufficiently high order of approximation can give accurate enough approximations so that no iteration is needed, as shown in Figures 9–12.

3.5. $\alpha = \frac{1}{2}$, $\beta = 1$ *and* $\mathscr{L} = \mathscr{L}_3$

Here we use $\mathscr{L}_3$ defined by (3.8) as the auxiliary linear operator. Our calculations indicate that, when $0 < \gamma \leq 50$ and $-10 \leq \hbar < 0$, formula (3.11) at a sufficiently high order of approximation can give accurate enough approximations so that no iteration is needed, as shown in Figures 13–16.

3.6. $\alpha = \beta = 3$ *and* $\mathscr{L} = \mathscr{L}_3$

Here we use $\mathscr{L}_3$ defined by (3.8) as the auxiliary linear operator. Our calculations indicate that, when $0 < \gamma \leq 20$ and $-4 \leq \hbar < 0$, formula (3.11) at a sufficiently high order of approximation can give accurate enough approximations so that no iteration is needed, as shown in Figures 17–20.

Using the above-mentioned three kinds of auxiliary linear operators, we calculated many cases for different values of α and β, and have found that for all of them, the proposed

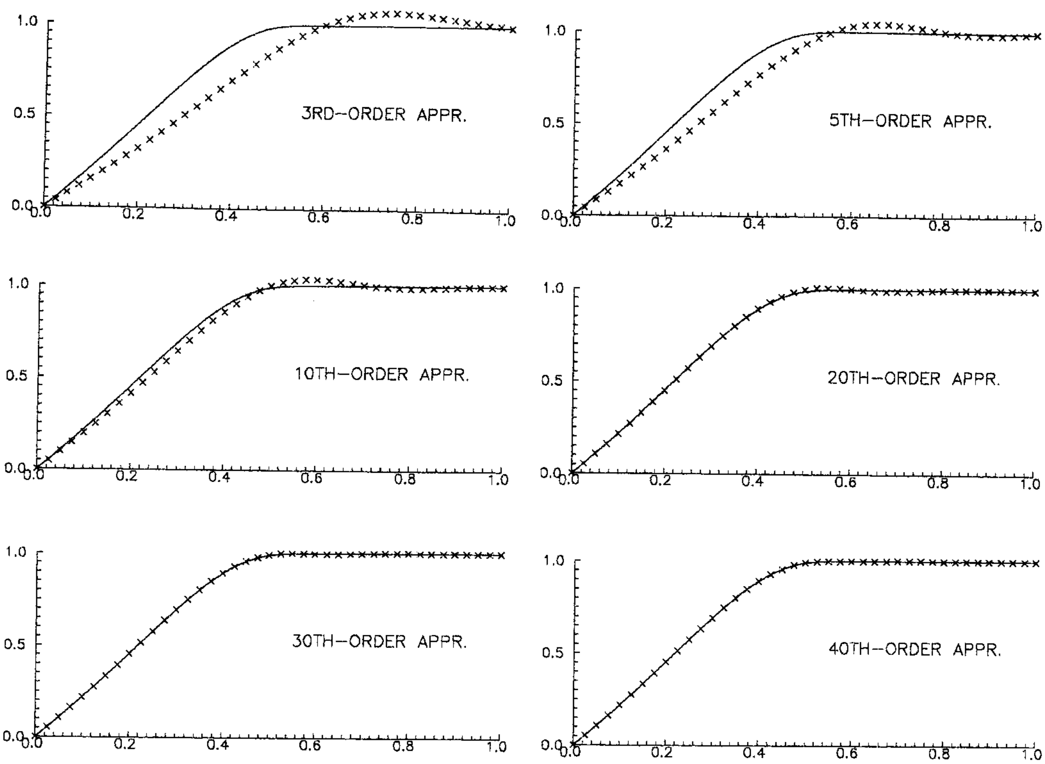

Figure 17. Approximations in the case of $\alpha = \beta = 3$, $\hbar = -4$ and $\mathscr{L} = \mathscr{L}_3$ ($\gamma = 20$); $\times$, approximate results by means of no iterations; —, exact solution; horizontal axis, τ; vertical axis, $\phi(\tau)$.

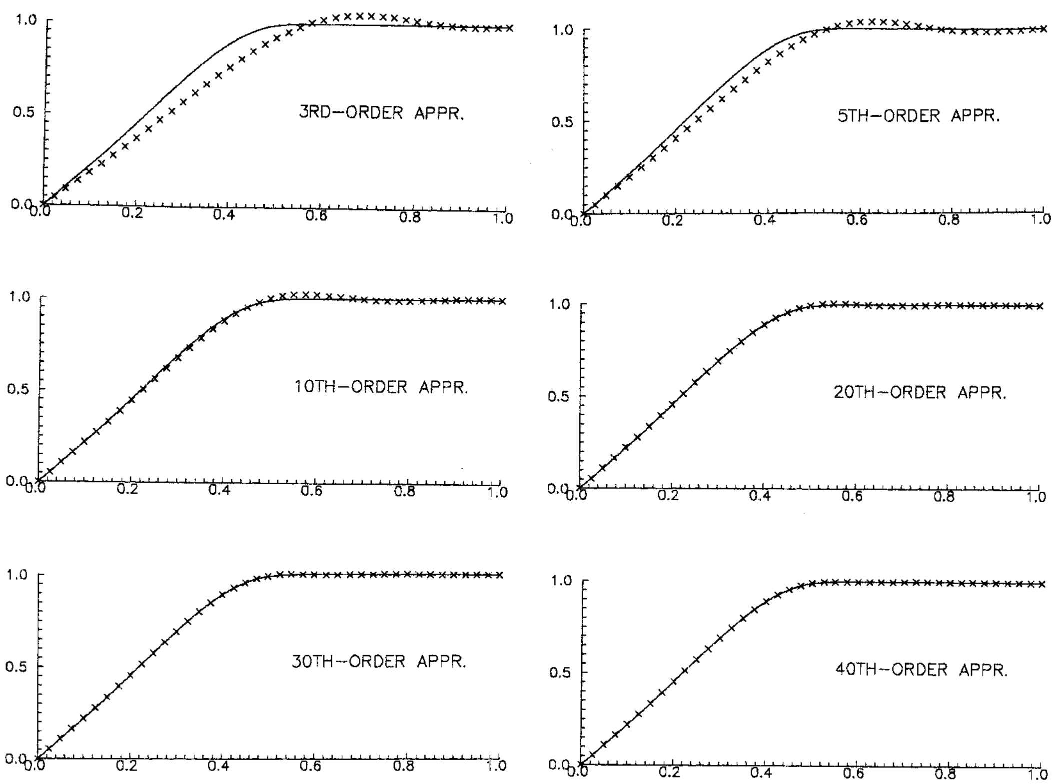

Figure 18. Approximations in the case of $\alpha = \beta = 3$, $\hbar = -4$ and $\mathscr{L} = \mathscr{L}_3$ ($\gamma = 10$); $\times$, approximate results by means of no iterations; —, exact solution; horizontal axis, τ; vertical axis, $\phi(\tau)$.

approach can give accurate enough approximations of the non-linear equations (2.4) and (2.5) without any iterations, if proper values of $\hbar$ and γ (when $\mathscr{L}_2$ or $\mathscr{L}_3$ is selected as the auxiliary linear operator) are used. These calculations indicate that for a given auxiliary linear operator, there is a region of $\hbar$ in which the proposed direct approach is valid. Notice that, when using $\mathscr{L}_2$ defined by (3.7) or $\mathscr{L}_3$ defined by (3.8) as auxiliary linear operator, the proposed approach is valid in the large regions of γ and $\hbar$, although $\mathscr{L}_2$ and $\mathscr{L}_3$ have nearly no relationship with the original non-linear equation (2.4). All of these verify that the proposed direct approach is not very sensitive to the selection of the auxiliary linear operator $\mathscr{L}$ and the value of $\hbar$. However, for a non-linear problem and a given auxiliary linear operator, we have not known how to determine the range of $\hbar$ to ensure that the proposed approach is valid in general.

Note that the approach is still valid if one applies other numerical techniques, such as the finite element method (FEM), the finite volume method (FVM), and so on, to solve the related linear equations (2.15) and (2.16), as mentioned by Liao [8]. Moreover, when we select $\mathscr{L}_2$ defined by (3.7) or $\mathscr{L}_3$ defined by (3.8) as the auxiliary linear operator whose fundamental solution is known, Equations (2.15) and (2.16) can be solved by the BEM, and in this case the proposed approach becomes the so-called general boundary element method proposed in [5–7]. Hence, the proposed non-iterative numerical approach to non-linear problems has rather general meanings.

Note that, here, the FDM was used to solve (2.15) and (2.16) and the related coefficient matrix of the discrete algebraic equations are the same for all $k \geq 1$ so that it is numerically efficient to solve all of the kth-order ($k \geq 1$) deformation equations (2.15) and (2.16). Similarly, one can apply the proposed approach to two- or three-dimensional non-linear problems, if one can find a sufficiently efficient numerical method to solve the related high-order deformation equations.

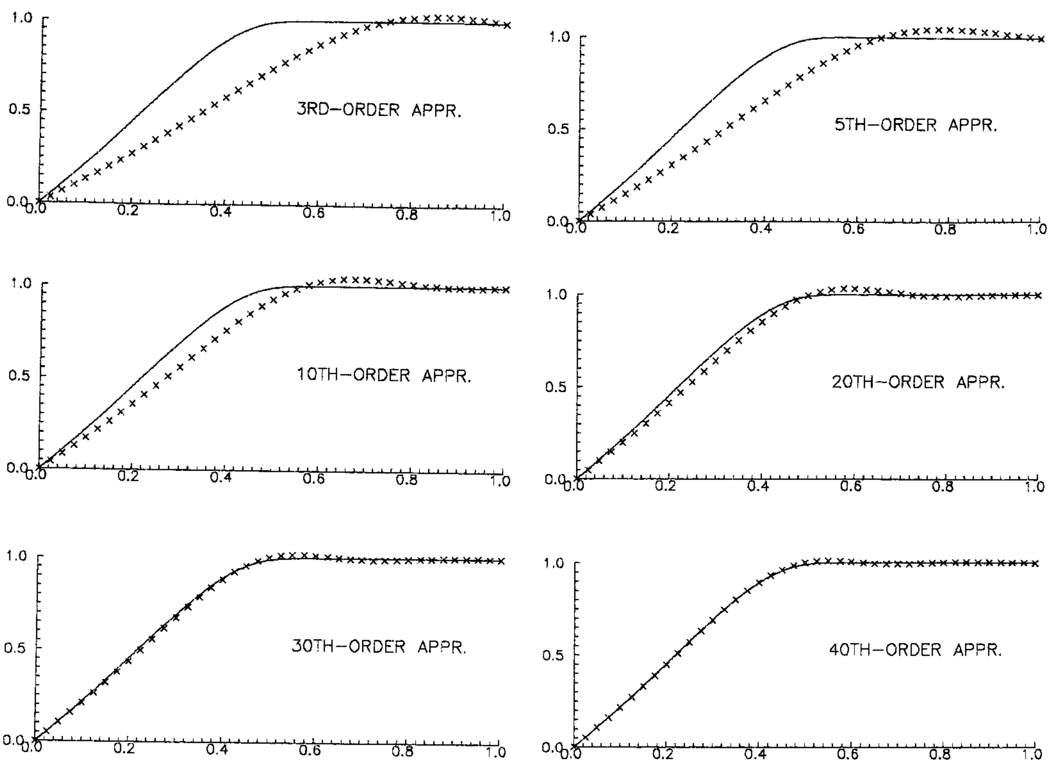

Figure 19. Approximations in the case of $\alpha = \beta = 3$, $\hbar = -2$ and $\mathscr{L} = \mathscr{L}_3$ ($\gamma = 20$); $\times$, approximate results by means of no iterations; —, exact solution; horizontal axis, τ; vertical axis, $\phi(\tau)$.

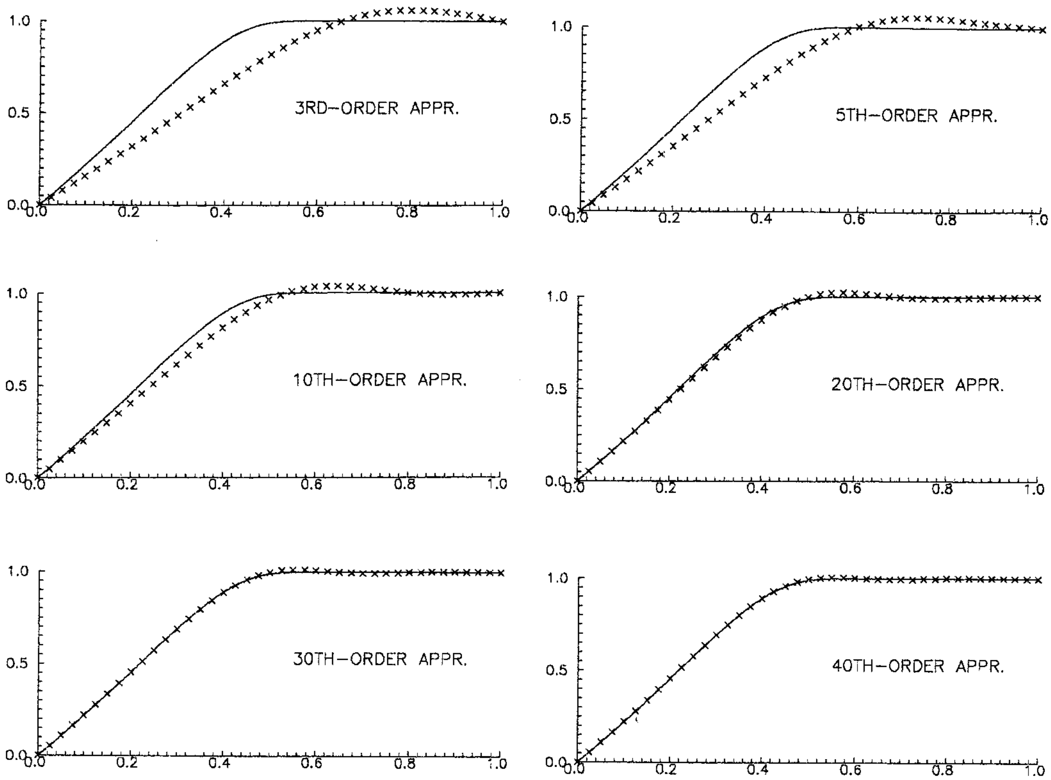

Figure 20. Approximations in the case of $\alpha = \beta = 3$, $\hbar = -2$ and $\mathscr{L} = \mathscr{L}_3$ ($\gamma = 10$); $\times$, approximate results by means of no iterations; —, exact solution; horizontal axis, τ; vertical axis, $\phi(\tau)$.

4. CONCLUSIONS AND DISCUSSIONS

In this paper, we propose a non-iterative (direct) numerical approach for two-dimensional viscous flow problems governed by the non-linear Falkner–Skan equation. We prove that, as long as the related infinite series (2.19) converges, it must converge to the solution of the original non-linear Equations (2.4) and (2.5). Our numerical calculations verify the validity of the proposed direct numerical approach.

This direct numerical method has the following significance. First of all, an accurate enough approximation of a strongly non-linear problem can be obtained by the proposed approach even with *no* iteration, if the parameter $\hbar$, the initial guess and the auxiliary linear operator $\mathscr{L}$ are properly selected. This might shake the governing place of iterative methodology in solving non-linear problems. Notice that, currently, many researchers apply direct methods to solve linear partial differential equations (for instance, Greengard and Lee [11], and McKenney and Greengard [12]). The proposed approach verifies that an effective, non-iterative numerical technique for non-linear problems can be given, and this might break some new grounds of research. Secondly, the proposed numerical approach logically includes the iterative methodology, because when the order M is low, Equation (3.11) also provides us with a family of iterative formulae in parameters $\hbar$ and M, and our numerical calculations verify once again that the higher the order of approximation, the faster the related iterative process converges; a similar conclusion to that given by Liao [8]. On the other hand, when the order M of approximation is high enough, no iteration is necessary, as shown in Figures 1 and 3 and Figures 5–20. Thus, the proposed numerical approach would seem to have more general meanings. Finally, it shows the possibility that if we could find the analytical solution of the linear kth-order deformation equations, we even might obtain the analytical approximations of the considered viscous flow. This might encourage us to apply the HAM to give some purely analytical solutions of some viscous flow problems, as performed by Liao [3,4].

Although we have considered here a simple problem, i.e. two-dimensional viscous flow, the non-linearity of (2.4) is rather strong. We think that the basic ideas of the proposed non-iterative numerical approach might be applied to solve other strongly non-linear problems in science and engineering.

ACKNOWLEDGMENTS

Thanks to reviewers for their helpful suggestions. This work is supported partly by the Natural Science Foundation of China (Approved No. 19702014) and partly by the Supporting Outstanding Young Teacher Foundation, Education Ministry of China.

APPENDIX A. NOMENCLATURE

$\mathscr{A}$	Non-linear differential operator for governing equation
$a(\tau)$, $b(\tau)$, $c(\tau)$	The real functions, as coefficients of the auxiliary linear operator (3.1)
$A(\tau)$, $B(\tau)$, $C(\tau)$	The real functions defined by Equations (3.3), (3.4) and (3.5) respectively
$\mathscr{E}_M$	r.m.s. residual error of (2.4) under the Mth-order approximation
$\hbar$	Non-zero auxiliary parameter
$\mathscr{L}$, $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$	Auxiliary linear operator
M	Order of approximation
p	Embedding parameter

Greek letters

α, β	Parameters in the Falkner–Skan equation (2.1)
$\phi(\tau)$	$=F'(\eta)$, dependent variable
γ	Parameter in auxiliary linear operators $\mathscr{L}_2$ and $\mathscr{L}_3$
η	Independent variable
τ	Independent variable defined by (2.3)
$\Phi(\tau, p, \hbar)$	Homotopy of $\phi(\tau)$

REFERENCES

1. Liao SJ. *A kind of linear invariance under homotopy and some simple applications of it in mechanics*. Bericht. No. 520, IFS, The University of Hamburg, 1992.
2. Liao SJ. A kind of approximate solution technique which does not depend upon small parameters (Part 2). *International Journal of Non-Linear Mechanics* 1997; **32**: 815–822.
3. Liao SJ. An explicit, totally analytic approximate solution for Blasius' viscous flow problems. *International Journal of Non-Linear Mechanics* 1999; **34**: 759–778.
4. Liao SJ. A uniformly valid analytic solution of two-dimensional viscous flow over a semi-infinite flat plate. *Journal of Fluid Mechanics* 1999; **385**: 101–128.
5. Liao SJ. High-order BEM formulations for strongly non-linear problems governed by quite general non-linear differential operators. *International Journal for Numerical Methods in Fluids* 1996; **23**: 739–751.
6. Liao SJ. On the general boundary element method. *Engineering Analysis with Boundary Elements* 1998; **21**: 39–51.
7. Liao SJ, Chwang AT. The general BEM for strongly nonlinear problems. *International Journal for Numerical Methods in Fluids* 1996; **23**: 467–483.
8. Liao SJ. Numerically solving non-linear problems by the homotopy analysis method. *Computational Mechanics* 1997; **20**: 530–540.
9. Blasius H. Grenzschichten in Fluessingkeiten mit kleiner Reibung. *Zeitschrift für Mathematik und Physik* 1908; **56**.
10. Falkner VM, Skan SW. Some approximate solutions of the boundary layer equations. *Philosophical Magazine* 1931; **12**: 865.
11. Greengrad L, Lee JY. A direct adaptive Poisson solver of arbitrary order accuracy. *Journal of Computational Physics* 1996; **125**: 415–424.
12. McKenney A, Greengard L. A fast Poisson solver of complex geometries. *Journal of Computational Physics* 1995; **118**: 348–355.